

THE BOUNDARY OF THE MODULI SPACE OF STABLE CUBIC FIVEFOLDS

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Abstract

By GIT theory due to Mumford, the moduli space of stable cubic fivefolds is compactified by adding non stable semi-stable (i.e. strictly semi-stable) locus. In this paper, we prove that this locus consists of 19 components. Moreover, we give a description of equation and singularity of cubic fivefold corresponding to the generic point in each component.

0 Introduction

A geometry of cubic fivefolds has several features. Hodge theoretically, for example, it has the intermediate Jacobian which is generically 21 dimensional principally polarized Abelian variety, and the Able-Jacobi map is generically isomorphism [2]. This phenomenon is similar to the case of cubic threefolds and algebraic curves. Though study of degeneration of cubic threefolds is basic subject, the structure of boundary of the moduli space of cubic fivefolds is quite complicated comparing to that of cubic threefolds. In this paper, we study boundary components of GIT compactification. Here GIT compactification of the moduli space of cubic hypersurfaces in \mathbb{P}^n means the categorical quotient $\mathbb{P}(\mathrm{Sym}_{n+1}^3)^{ss} // \mathrm{SL}(n+1)$, where Sym_{n+1}^3 is the vector space of homogeneous polynomials of degree 3 of $n+1$ variables.

Let us recall some previous work on the GIT compactifications of moduli spaces of cubic hypersurfaces. David Hilbert studied the case of cubic surfaces [6] and the main result is the following:

Theorem 0.1. *Let S be a cubic surface in \mathbb{P}^3 . Then,*

- *S is stable if and only if it has only rational double points of type A_1 .*
- *S is semi-stable if and only if it has only rational double points of type A_1 or A_2 .*
- *The moduli space of stable cubic surfaces is compactified by adding one point corresponding to the semi-stable cubic $x_0x_1x_2 + x_3^3 = 0$ with 3 A_2 singularities.*

Mutsumi Yokoyama studied the case of cubic threefolds [9]. The main result is as follows:

Theorem 0.2. *Let X be a cubic threefold in \mathbb{P}^4 . Then,*

- *X is stable if and only if it has only double points of type A_n with $n \leq 4$.*
- *X is semi-stable if and only if it has only double points of type A_n with $n \leq 5$, D_4 , or A_∞ .*
- *The moduli space of stable cubic threefolds is compactified by adding two components corresponding to strictly semi-stable cubic threefolds. One is isomorphic to \mathbb{P}^1 and the other is an isolated point corresponding to the semi-stable cubic threefold $x_0x_1x_2 + x_3^3 + x_4^3 = 0$ with 3 D_4 singularities.*

M.Yokoyama and Radu Laza studied the case of cubic fourfold [10],[7]. They proved the following theorems.

Theorem 0.3. *A strictly semi-stable cubic fourfold is projectively equivalent to the following 6 types.*

- $f_1 = c(x_0, x_1, x_2) + q_1(x_0, x_1, x_2)x_3 + q_2(x_0, x_1, x_2)x_4 + q_3(x_0, x_1, x_2)x_5 + x_0q_4(x_3, x_4, x_5)$
- $f_2 = c(x_0, \dots, x_3) + q_1(x_0, \dots, x_3)x_4 + q_2(x_0, \dots, x_3)x_5$
- $f_3 = c(x_0, x_1) + q_1(x_0, x_1)l_1(x_2, x_3, x_4, x_5) + l_2(x_0, x_1)q_2(x_2, x_3, x_4, x_5)$
- $f_4 = c(x_0, \dots, x_4) + q(x_0, x_1)x_5$
- $f_5 = c(x_0, \dots, x_3) + (q_1(x_0, x_1) + l_1(x_0, x_1)x_2 + l_2(x_0, x_1)x_3)x_4 + (q_2(x_0, x_1) + x_0l_3(x_2, x_3))x_5 + \alpha x_0x_4^2$
- $f_6 = c(x_0, x_1, x_2) + q_1(x_0, x_1, x_2)x_3 + q_2(x_0, x_1, x_2)x_4 + q_3(x_0, x_1, x_2)x_5 + l_1(x_0, x_1, x_2)x_3^2 + l_2(x_0, x_1, x_2)x_3x_4$

Here, l, q, c, α stand for linear forms, quadratic forms, cubic forms, and constant term respectively.

Theorem 0.4. *The singular locus of a strictly semi-stable cubic fourfold contains one of the following:*

- *a point*
- *a line*
- *a conic*
- *a $(2, 2)$ -intersection in \mathbb{P}^3*

In this paper, we study the space $\mathbb{P}(\text{Sym}_7^3)^{ss}/\text{SL}(7)$, i.e. the GIT compactification of the moduli space of cubic fivefolds. In particular, we investigate the boundary of the moduli space of the stable cubic fivefolds. i.e. strictly semi-stable cubic fivefolds.

The main theorem of this paper is the following two Theorems. We present an algorithm which is executable by a computer.

Theorem 0.5. *The strictly semi-stable locus of the moduli space of cubic fivefolds consists of 19 irreducible components. Moreover, the polynomials corresponding to the generic points of the 19 irreducible components are given as f_i ($i = 1, \dots, 22$, $i \neq 14, 15, 22$) listed in Theorem 3.2.*

Theorem 0.6. *The singular locus of a strictly semi-stable cubic fivefold contains one of the following:*

- *a point whose multiplicity is equals or greater than 15.*
- *a line whose multiplicity is 1 or 2.*
- *two lines which intersect at one point whose multiplicity is 1.*
- *a conic whose multiplicity is 1 or 2.*
- *a $(2, 2)$ -intersection in \mathbb{P}^3 whose multiplicity is 1 or 2.*

This paper consists of 6 sections. In section 1, we review the numerical criterion of stability. In section 2, we obtain 22 families of strictly semi-stable cubic fivefolds with respect to a fixed maximal torus. In section 3, we study the singular loci of the above 22 families. In section 4, we study inclusions among these 22 families under the action of $\mathrm{SL}(7)$. In section 5, we summarize the results.

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1 The numerical criterion for cubic fivefolds

In this section, we review the numerical criterion for stability or semi-stability of cubic fivefolds. We use the following notations.

- Let $\mathbb{C}[x_0, \dots, x_6]_3$ be the set of homogeneous polynomials of degree 3.
- For a vector $\mathbf{x} \in \mathbb{Q}^7$, $\mathrm{wt}(\mathbf{x}) = \sum_{k=0}^6 x_k$ is called the wight of \mathbf{x} .
- We define $\mathbb{Z}_{\geq 0}^7 = \{\mathbf{x} = (x_0, x_1, \dots, x_6) \in \mathbb{Z}^7 \mid x_k \geq 0 (k = 0, 1, \dots, 6)\}$,

$$\mathbb{Z}_{(d)}^7 = \{\mathbf{x} \in \mathbb{Z}^7 \mid \mathrm{wt}(\mathbf{x}) = d\},$$

$$\mathbb{I} = \mathbb{Z}_{(3)}^7 \cap \mathbb{Z}_{\geq 0}^7 \text{ and it is simply called the simplex.}$$

- For $\mathbf{r} \in \mathbb{Q}^7$, we define $\mathbb{I}(\mathbf{r})_{\geq 0} = \{\mathbf{i} \in \mathbb{I} \mid \mathbf{r} \cdot \mathbf{i} \geq 0\}$ and $\mathbb{I}(\mathbf{r})_{> 0} = \{\mathbf{i} \in \mathbb{I} \mid \mathbf{r} \cdot \mathbf{i} > 0\}$, here \cdot denotes the standard inner product of vectors.
- For a polynomial $f = \sum_{\mathrm{wt}(\mathbf{i})=3} a_{\mathbf{i}} x^{\mathbf{i}} \in \mathbb{C}[x_0, \dots, x_6]_3$, we define the support of f by $\mathrm{Supp}(f) = \{\mathbf{i} \in \mathbb{I} \mid a_{\mathbf{i}} \neq 0\}$

- We set $\eta = (3/7, 3/7, 3/7, 3/7, 3/7, 3/7, 3/7) \in \mathbb{Q}^7$ and it is called the barycenter of the simplex \mathbb{I} .
- A vector $\mathbf{r} \in \mathbb{Z}^7$ is said to be reduced when there is no integer α such that $|\alpha| \geq 2$ and $\frac{1}{\alpha}\mathbf{r} \in \mathbb{Z}^7$

We fix a maximal torus \mathbb{T} of $\mathrm{SL}(7)$. Consider a one parameter subgroup (1-PS for short) $\lambda : \mathbb{G}_m \rightarrow \mathrm{SL}(7)$ whose image is contained in \mathbb{T} . For suitable basis of \mathbb{C}^7 , λ can be expressed as a diagonal matrix $\mathrm{diag}(t^{r_0}, t^{r_1}, \dots, t^{r_6})$ where $t \neq 0$ is a parameter of \mathbb{G}_m . Let us choose and fix such basis. Then λ corresponds to an element $\mathbf{r} = (r_0, r_1, \dots, r_6)$ in $\mathbb{Z}_{(0)}^7$. We can regard an element of $\mathbb{Z}_{(0)}^7$ as a 1-PS of \mathbb{T} .

Definition 1.1. Let s be a subset of \mathbb{I} . We say that s is not stable (resp. unstable) with respect to \mathbb{T} when $s \subseteq \mathbb{I}(\mathbf{r})_{\geq 0}$ (resp. $s \subseteq \mathbb{I}(\mathbf{r})_{> 0}$) for some 1-PS \mathbf{r} . For $0 \neq f \in \mathbb{C}[x_0, \dots, x_6]_3$, we say that f is not stable (resp. unstable) with respect to \mathbb{T} when $\mathrm{Supp}(f) \subseteq \mathbb{I}$ is not stable (resp. unstable) with respect to \mathbb{T} . See Figure 1 for more details.

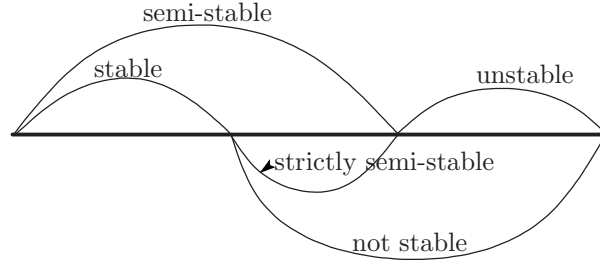


Figure 1: various concepts of stability

The following theorem is the numerical criterion for stability via the language of convex geometry.

Theorem 1.2. The cubic fivefold defined by $f \in \mathbb{C}[x_0, \dots, x_6]_3$ is not stable (resp. unstable) if and only if there exists an element $\sigma \in \mathrm{SL}(7)$ such that f^σ is not stable (resp. unstable) with respect to \mathbb{T} .

In particular, f is strictly semi-stable if and only if

- (1) There exist $\sigma \in \mathrm{SL}(7)$ such that f^σ is not stable with respect to \mathbb{T} , and
- (2) For any $\sigma \in \mathrm{SL}(7)$, f^σ is semi-stable with respect to \mathbb{T} .

Proof. See Theorem 9.3 of [4]. □

2 The maximal cubic fivefolds which is strictly semi-stable with respect to the maximal torus \mathbb{T}

In this paper, we list up the irreducible components corresponding to strictly semi-stable cubic fivefolds. For this purpose, we list up all strictly semi-stable cubic fivefolds with respect to the maximal torus \mathbb{T} . To solve this problem, we will consider the set of maximal strictly semi-stable subset of \mathbb{I} . The order in the set of subsets of \mathbb{I} is given by inclusion. For this purpose, we list up the set of all maximal elements of $\mathcal{S} = \{\mathbb{I}(\mathbf{r})_{\geq 0} \mid \mathbf{r} \in \mathbb{Z}_{(0)}^7\}$.

We solve this problem using computer. We need an algorithm which enable us to obtain them in finite steps. Before giving such an algorithm, we remark that $\mathbb{I}(\mathbf{r})_{\geq 0}$ and $\mathbb{I}(\mathbf{r}')_{\geq 0}$ might be same for two different vectors $\mathbf{r}, \mathbf{r}' \in \mathbb{Z}_{(0)}^7$.

Lemma 2.1. *Let $\mathbb{I}(\mathbf{r})_{\geq 0}$ be a maximal element of \mathcal{S} , where $\mathbf{r} \in \mathbb{Z}_{(0)}^7$. Then there exist 5 elements $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_5 \in \mathbb{I}$ and a vector $\mathbf{r}' \in \mathbb{Z}_{(0)}^7$ such that they satisfy the following three conditions:*

- (1) *The vector subspace W of \mathbb{Q}^7 spanned by $\mathbf{x}_1, \dots, \mathbf{x}_5, \eta$ over \mathbb{Q} has dimension 6*
- (2) *The vector \mathbf{r}' is orthogonal to the subspace W of \mathbb{Q}^7 .*
- (3) $\mathbb{I}(\mathbf{r})_{\geq 0} = \mathbb{I}(\mathbf{r}')_{\geq 0}$

Proof. Let us put $C = \mathbb{I}(\mathbf{r}) \cup \eta$. We consider the convex hull \check{C} of C in \mathbb{Q}^7 . Let F be a face of \check{C} containing the point η . There is a normal vector \mathbf{r}' of F in $\mathbb{Z}_{(0)}^7$ such that $\check{C} \subseteq \{\mathbf{x} \in \mathbb{Q}^7 \mid \mathbf{r}' \cdot \mathbf{x} \geq 0\}$. We have $\text{wt}(\mathbf{r}') = 0$ since the hyperplane defined by $\{\mathbf{x} \in \mathbb{Q}^7 \mid \mathbf{r}' \cdot \mathbf{x} = 0\}$ passes through the point η . By the definition of the faces of a convex set in \mathbb{Q}^7 , we can take 5 points $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_5$ from the set $\mathbb{I} \cap F$ such that $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_5, \eta$ are linearly independent over \mathbb{Q} . In general we have $\mathbb{I}(\mathbf{r})_{\geq 0} \subseteq \mathbb{I}(\mathbf{r}')_{\geq 0}$, and by the assumption that $\mathbb{I}(\mathbf{r})_{\geq 0}$ is maximal in \mathcal{S} , we conclude that $\mathbb{I}(\mathbf{r})_{\geq 0} = \mathbb{I}(\mathbf{r}')_{\geq 0}$. \square

By this lemma, we can determine the set of maximal elements of \mathcal{S} up to permutations of coordinates in finite steps using the following algorithm.

Algorithm 2.2. *Let \mathcal{F} be the set of five different points of \mathbb{I} . We fix a total order on \mathcal{F} . As an initial data, we set $\mathcal{S}' = \emptyset$ and $\mathbf{x} = (x_0, \dots, x_5)$ be the minimum element of \mathcal{F} . We will modify \mathcal{S}' using the following algorithm.*

- *Step 1. If the subspace W spanned by x_0, \dots, x_5, η of \mathbb{Q}^7 has dimension 6 then take a reduced normal vector $\mathbf{r} = (r_0, \dots, r_6) \in \mathbb{Z}_{(0)}^7$ of W and go to step2, else go to step5.*
- *Step 2. If $\mathbf{r} = (r_0, \dots, r_6)$ satisfy the condition $r_0 \geq \dots \geq r_6$ or $r_0 \leq \dots \leq r_6$, then go to step3, else go to step5.*

- Step 3. If $r_0 \geq \dots \geq r_6$ (resp. $r_0 \leq \dots \leq r_6$) add $\mathbb{I}(\mathbf{r})$ (resp. $\mathbb{I}(-\mathbf{r})$) to \mathcal{S}' and go to step 4.
- Step 4. Delete all elements of \mathcal{S}' which is not maximal in \mathcal{S}' and go to step 5.
- Step 5. We replace the element \mathbf{x} by next element if \mathbf{x} is not the maximum element and go to step 1. Otherwise we stop the algorithm.

We remark step 2 kills the symmetry S_7 action on the variables x_0, \dots, x_6 . We also remark step 4 is not essential but technical to save the memory of a computer. After running this algorithm with the aid of computer, we find 23 elements $\mathbb{I}(\mathbf{r}_1)_{\geq 0}, \dots, \mathbb{I}(\mathbf{r}_{23})_{\geq 0}$ in \mathcal{S}' , where $\mathbf{r}_k = (r_0, \dots, r_6) \in \mathbb{Z}_{(0)}^7$ is a reduced vector with $r_0 \geq \dots \geq r_6$. When we compute the convex hulls of $\mathbb{I}(\mathbf{r}_1)_{\geq 0}, \dots, \mathbb{I}(\mathbf{r}_{23})_{\geq 0}$ in \mathbb{Q}^7 , then only one of the convex hull of $\mathbb{I}(\mathbf{r}_k)_{\geq 0}$ does not contain η [5]. Let us call it $\mathbb{I}(\mathbf{r}_{23})_{\geq 0}$. Because only $\mathbb{I}(\mathbf{r}_{23})_{\geq 0}$ is unstable with respect to \mathbb{T} , we do not treat it when it does not need. So we can conclude that there are 22 maximal strictly semi-stable cubic fivefolds for the fixed maximal torus \mathbb{T} . As a consequence of this algorithm, we have the following proposition.

Proposition 2.3. *The set $\mathcal{M} = \{\mathbb{I}(\mathbf{r}_1)_{\geq 0}, \dots, \mathbb{I}(\mathbf{r}_{22})_{\geq 0}\}$ is given as follows.*

$\mathbf{r}_1 = (8, 3, 2, -1, -2, -4, -6)$	$\mathbf{r}_2 = (6, 4, 1, -1, -2, -3, -5)$
$\mathbf{r}_3 = (4, 2, 1, -1, -1, -2, -3)$	$\mathbf{r}_4 = (2, 2, 0, -1, -1, -1, -1)$
$\mathbf{r}_5 = (3, 2, 1, 0, -1, -2, -3)$	$\mathbf{r}_6 = (4, 2, 1, 0, -1, -2, -4)$
$\mathbf{r}_7 = (5, 3, 2, 1, -1, -4, -6)$	$\mathbf{r}_8 = (6, 4, 2, 1, -2, -3, -8)$
$\mathbf{r}_9 = (4, 1, 1, 0, -2, -2, -2)$	$\mathbf{r}_{10} = (2, 2, 0, 0, -1, -1, -2)$
$\mathbf{r}_{11} = (2, 1, 0, 0, -1, -1, -1)$	$\mathbf{r}_{12} = (3, 2, 1, 1, -1, -2, -4)$
$\mathbf{r}_{13} = (2, 1, 1, 0, -1, -1, -2)$	$\mathbf{r}_{14} = (2, 2, 2, 0, -1, -1, -4)$
$\mathbf{r}_{15} = (2, 0, 0, 0, 0, -1, -1)$	$\mathbf{r}_{16} = (2, 1, 1, 0, 0, -2, -2)$
$\mathbf{r}_{17} = (2, 1, 0, 0, 0, -1, -2)$	$\mathbf{r}_{18} = (1, 1, 1, 0, 0, -1, -2)$
$\mathbf{r}_{19} = (1, 1, 0, 0, 0, -1, -1)$	$\mathbf{r}_{20} = (1, 1, 1, 1, 0, -2, -2)$
$\mathbf{r}_{21} = (1, 1, 0, 0, 0, 0, -2)$	$\mathbf{r}_{22} = (1, 0, 0, 0, 0, 0, -1)$

For example, $\mathbb{I}(\mathbf{r}_1)_{\geq 0}$ is
 $\mathbb{I}(\mathbf{r}_1)_{\geq 0} = \{x_0^3, x_0^2x_1, x_0^2x_2, x_0^2x_3, x_0^2x_4, x_0^2x_5, x_0^2x_6, x_0x_1^2, x_0x_1x_2, x_0x_1x_3, x_0x_1x_4, x_0x_1x_5, x_0x_1x_6, x_0x_2^2, x_0x_2x_3, x_0x_2x_4, x_0x_2x_5, x_0x_2x_6, x_0x_3^2, x_0x_3x_4, x_0x_3x_5, x_0x_3x_6, x_0x_4^2, x_0x_4x_5, x_0x_4x_6, x_0x_5^2, x_1^3, x_1^2x_2, x_1^2x_3, x_1^2x_4, x_1^2x_5, x_1^2x_6, x_1x_2^2, x_1x_2x_3, x_1x_2x_4, x_1x_2x_5, x_1x_2x_6, x_1x_3^2, x_1x_3x_4, x_1x_3x_5, x_1x_3x_6, x_1x_4^2, x_1x_4x_5, x_1x_4x_6, x_1x_5^2, x_1x_5x_6, x_1x_6^2, x_2^3, x_2^2x_3, x_2^2x_4, x_2^2x_5, x_2^2x_6, x_2x_3^2, x_2x_3x_4, x_2x_3x_5, x_2x_3x_6, x_2x_4^2, x_2x_4x_5, x_2x_4x_6, x_2x_5^2, x_2x_5x_6, x_2x_6^2, x_3^3, x_3^2x_4, x_3^2x_5, x_3^2x_6, x_3x_4^2, x_3x_4x_5, x_3x_4x_6, x_3x_5^2, x_3x_5x_6, x_3x_6^2, x_4^3, x_4^2x_5, x_4^2x_6, x_4x_5^2, x_4x_5x_6, x_4x_6^2, x_5^3, x_5^2x_6, x_5x_6^2, x_6^3\}$.
Here we use the notation $x_0^{i_0}x_1^{i_1}\dots x_6^{i_6}$ for an element $(i_0, i_1, \dots, i_6) \in \mathbb{Z}_{(3)}^7$ in order to save the space.

Remark 2.4. *The following vectors can be \mathbf{r}_{23} . i.e. there are several vectors which give the set $\mathbb{I}(\mathbf{r}_{23})_{\geq 0}$.*

$(8, 5, 3, 2, -4, -4, -10)$	$(8, 5, 2, 2, -3, -4, -10)$
$(6, 4, 2, 2, -3, -3, -8)$	$(5, 3, 1, 1, -2, -2, -6)$
$(4, 3, 1, 1, -2, -2, -5)$	$(4, 2, 1, 1, -2, -2, -4)$

Here, we define $\mathbf{r}_{23} = (8, 5, 3, 2, -4, -4, -10)$.

3 The singular loci of the above 22 cubic fivefolds

An element $\mathbb{I}(\mathbf{r}_k)_{\geq 0}$ of \mathcal{M} represents a family of cubic fivefold whose defining polynomial's support are contained in $\mathbb{I}(\mathbf{r}_k)_{\geq 0}$. In this section, we investigate the singular locus of a generic cubic fivefold of the family $\mathbb{I}(\mathbf{r}_k)_{\geq 0}$. Let f_k be a generic polynomial whose support is $\mathbb{I}(\mathbf{r}_k)_{\geq 0}$. ($k = 1, 2, \dots, 22$). Since any smooth cubic fivefold is stable, the variety $V(f_k)$ defined by f_k should have singular points. If we express f_k directly it becomes too long, so we prepare a notation.

Definition 3.1. *The symbols c, q, l, α stand for a cubic form, a quadratic form, a liner form, a constant term respectively. Similarly the symbols q_i, l_i, α_i stand for i -th quadratic form, a liner form, a constant term respectively.*

The following theorem is a direct consequence of the list in proposition 2.3.

Theorem 3.2. *Using the above notations, the generic polynomials of f_1, \dots, f_{22} are the following forms.*

- $f_1 = c(x_0, x_1, x_2) + q_1(x_0, x_1, x_2)x_3 + l_1(x_0, x_1, x_2)x_3^2 + (q_2(x_0, x_1, x_2) + l_2(x_0, x_1)x_3)x_4 + (q_3(x_0, x_1, x_2) + x_0x_3)x_5 + (q_4(x_0, x_1) + x_0x_2 + x_0x_3)x_6 + x_0(q_5(x_4, x_5) + x_4x_6)$
- $f_2 = c(x_0, x_1, x_2) + q_1(x_0, x_1, x_2)x_3 + q_2(x_0, x_1, x_2)x_4 + (q_3(x_0, x_1) + l_1(x_0, x_1)x_2)x_5 + (q_4(x_0, x_1) + l_2(x_0, x_1)x_2)x_6 + x_0q_5(x_3, x_4, x_5) + x_1(q_6(x_3, x_4) + \alpha x_3x_5)$
- $f_3 = c(x_0, x_1, x_2) + q_1(x_0, x_1, x_2)x_3 + q_2(x_0, x_1, x_2)x_4 + q_3(x_0, x_1, x_2)x_5 + (q_4(x_0, x_1) + l(x_0, x_1)x_2)x_6 + x_0q_5(x_3, x_4, x_5) + x_1q_6(x_3, x_4)$
- $f_4 = c(x_0, x_1, x_2) + (q_1(x_0, x_1) + \alpha_1x_1x_2)x_3 + (q_2(x_0, x_1) + \alpha_2x_1x_2)x_4 + (q_3(x_0, x_1) + \alpha_3x_1x_2)x_5 + (q_4(x_0, x_1) + \alpha_4x_1x_2)x_6 + x_0q_5(x_3, \dots, x_6) + x_1q_6(x_3, \dots, x_6)$
- $f_5 = c(x_0, \dots, x_3) + (q_1(x_0, x_1, x_2) + l_1(x_0, x_1, x_2)x_3)x_4 + (q_2(x_0, x_1, x_2) + \alpha_1x_0x_3)x_5 + (q_3(x_0, x_1) + l_2(x_0, x_1)x_2 + \alpha_2x_0x_3)x_6 + x_0(\alpha_3x_4^2 + \alpha_4x_4x_5) + \alpha_5x_1x_4^2$
- $f_6 = c(x_0, \dots, x_3) + (q_1(x_0, x_1, x_2) + l_1(x_0, x_1, x_2)x_3)x_4 + (q_2(x_0, x_1, x_2) + l_2(x_0, x_1)x_3)x_5 + (q_3(x_0, x_1) + x_0l_3(x_2, x_3))x_6 + x_0q_4(x_4, x_5) + \alpha x_1x_4^2$
- $f_7 = c(x_0, \dots, x_3) + q_1(x_0, \dots, x_3)x_4 + (q_2(x_0, x_1, x_2) + l_1(x_0, x_1)x_3)x_5 + (q_3(x_0, x_1) + x_0l_2(x_2, x_3))x_6 + x_0(\alpha_1x_4^2 + \alpha_2x_4x_5) + \alpha_3x_1x_4^2 + \alpha_4x_2x_4^2$
- $f_8 = c(x_0, \dots, x_3) + q_1(x_0, \dots, x_3)x_4 + (q_2(x_0, x_1, x_2) + l(x_0, x_1, x_2)x_3)x_5 + q_3(x_0, x_1, x_2)x_6 + x_0q_4^{(2)}(x_4, x_5) + \alpha x_1x_4^2$
- $f_9 = c(x_0, \dots, x_3) + q_1(x_0, x_1, x_2)x_4 + q_2(x_0, x_1, x_2)x_5 + q_3(x_0, x_1, x_2)x_6 + x_0q_4(x_4, x_5, x_6)$

- $f_{10} = c(x_0, \dots, x_3) + (q_1(x_0, x_1) + x_0 l_1(x_2, x_3) + x_1 l_2(x_2, x_3))x_4 + (q_2(x_0, x_1) + x_0 l_3(x_2, x_3) + x_1 l_4(x_2, x_3))x_5 + (q_3(x_0, x_1) + x_0 l_5(x_2, x_3) + x_1 l_6(x_2, x_3))x_6 + x_0 q_4(x_4, x_5) + x_1 q_5(x_4, x_5)$
- $f_{11} = c(x_0, \dots, x_3) + (q_1(x_0, x_1) + l_1(x_0, x_1)x_2 + l_2(x_0, x_1)x_3)x_4 + (q_2(x_0, x_1) + l_3(x_0, x_1)x_2 + l_4(x_0, x_1)x_3)x_5 + (q_3(x_0, x_1) + l_5(x_0, x_1)x_2 + l_6(x_0, x_1)x_3)x_6 + x_0 q_4(x_4, x_5, x_6)$
- $f_{12} = c(x_0, \dots, x_3) + q_1(x_0, \dots, x_3)x_4 + q_2(x_0, \dots, x_3)x_5 + (q_3(x_0, x_1) + x_0 l(x_2, x_3))x_6 + x_0(\alpha_1 x_4^2 + \alpha_2 x_4 x_5) + \alpha_3 x_1 x_4^2$
- $f_{13} = c(x_0, \dots, x_3) + (q_1(x_0, x_1, x_2) + l_1(x_0, x_1, x_2)x_3)x_4 + (q_2(x_0, x_1, x_2) + l_2(x_0, x_1, x_2)x_3)x_5 + q_3(x_0, x_1, x_2)x_6 + x_0 q_4(x_4, x_5)$
- $f_{14} = c(x_0, \dots, x_3) + (q_1(x_0, \dots, x_2) + l_1(x_0, x_1, x_2)x_3)x_4 + (q_2(x_0, \dots, x_3) + l_2(x_0, x_1, x_2)x_3)x_5 + q_3(x_0, x_1, x_2)x_6 + x_0 q_4(x_4, x_5) + x_1 q_5(x_4, x_5) + x_2 q_6(x_4, x_5)$
- $f_{15} = c(x_0, \dots, x_4) + x_0 \{l_1(x_0, \dots, x_4)x_5 + l_2(x_0, \dots, x_4)x_6 + q(x_5, x_6)\}$
- $f_{16} = c(x_0, \dots, x_4) + (q_1^{(3)}(x_0, x_1, x_2) + \alpha_1 x_0 x_3 + \alpha_2 x_0 x_4)x_5 + (q_2(x_0, x_1, x_2) + \alpha_3 x_0 x_3 + \alpha_4 x_0 x_4)x_6$
- $f_{17} = c(x_0, \dots, x_4) + (q_1(x_0, x_1) + x_0 l_1(x_2, x_3, x_4) + x_1 l_2(x_2, x_3, x_4))x_5 + (q_2(x_0, x_1) + x_0 l_3(x_2, x_3, x_4))x_6 + \alpha x_0 x_5^2$
- $f_{18} = c(x_0, \dots, x_4) + (q_1(x_0, x_1, x_2) + l_1(x_0, x_1, x_2)x_3 + l_2(x_0, x_1, x_2)x_4)x_5 + q_2(x_0, x_1, x_2)x_6$
- $f_{19} = c(x_0, \dots, x_4) + (q_1(x_0, x_1) + x_0 l_1(x_2, x_3, x_4) + x_1 l_2(x_2, x_3, x_4))x_5 + (q_2(x_0, x_1) + x_0 l_3(x_2, x_3, x_4) + x_1 l_4(x_2, x_3, x_4))x_6$
- $f_{20} = c(x_0, \dots, x_4) + q_1(x_0, \dots, x_3)x_5 + q_2(x_0, \dots, x_3)x_6$
- $f_{21} = c(x_0, \dots, x_5) + q(x_0, x_1)x_6$
- $f_{22} = c(x_0, \dots, x_5) + x_0 l(x_0, \dots, x_5)x_6$
- $f_{23} = c(x_0, \dots, x_3) + q_1(x_0, \dots, x_3)x_4 + q_2(x_0, \dots, x_3)x_5 + (q_3(x_0, x_1) + x_0 l(x_2, x_3))x_6 + x_0 q_4(x_4, x_5)$

Next we compute the singular locus of f_k . We obtain the following list of the singular loci of the generic polynomials f_k . In the list, dim means dimension of the singular locus, degree means the degree of the Hilbert polynomial of the Jacobian ideal $(f_k, \frac{\partial f_k}{\partial x_0}, \dots, \frac{\partial f_k}{\partial x_6})$. We remark that for the Hilbert polynomial $p(t) = d/r! \cdot t^r + (\text{lower degree of } t)$ of a variety, d is the degree and r is the dimension of the variety. By the Euler's formula $3f_k = x_0 \frac{\partial f_k}{\partial x_0} + \dots + x_6 \frac{\partial f_k}{\partial x_6}$, it is enough to compute $(\frac{\partial f_k}{\partial x_0}, \dots, \frac{\partial f_k}{\partial x_6})$. Using Groebner basis, we can calculate the support and the Hilbert polynomial of the variety of this ideal [3].

The list of singular loci of f_k is as follows.

polynomial	singular locus	dimension	degree
f_1	a conic	1	2
f_2	a $(2, 2)$ -intersection in \mathbb{P}^3	1	4
f_3	a $(2, 2)$ -intersection in \mathbb{P}^3	1	4
f_4	a $(2, 2)$ -intersection in \mathbb{P}^3	1	8
f_5	a line	1	1
f_6	a point	0	19
f_7	a line	1	1
f_8	a point	0	17
f_9	a conic	1	4
f_{10}	a point	0	23
f_{11}	a conic	1	4
f_{12}	a line	1	1
f_{13}	two lines which intersect at one point	1	2
f_{14}	a point	0	16
f_{15}	two points	0	32
f_{16}	a line	1	2
f_{17}	a point	0	18
f_{18}	a line	1	1
f_{19}	a line	1	2
f_{20}	a line	1	2
f_{21}	a point	0	15
f_{22}	a point	0	15

Here we give the list of the equations of singular loci.

polynomial	equations
f_1	$q_5(x_4, x_5) + \alpha x_4 x_6 = 0, x_0 = x_1 = x_2 = x_3 = 0$
f_2	$q_5(x_3, x_4, x_5) = q_6(x_3, x_4) + \alpha x_3 x_5 = 0, x_0 = x_1 = x_2 = 0$
f_3	$q_5(x_3, x_4, x_5) = q_6(x_3, x_4) = 0, x_0 = x_1 = x_2 = 0$
f_4	$q_5(x_3, \dots, x_6) = q_6(x_3, \dots, x_6) = 0, x_0 = x_1 = x_2 = 0$
f_5	$x_0 = x_1 = x_2 = x_3 = x_4 = 0$
f_6	$x_0 = x_1 = x_2 = x_3 = x_4 = x_5 = 0$
f_7	$x_0 = x_1 = x_2 = x_3 = x_4 = 0$
f_8	$x_0 = x_1 = x_2 = x_3 = x_4 = x_5 = 0$
f_9	$q_4(x_4, x_5, x_6) = 0, x_0 = x_1 = x_2 = x_3 = 0$
f_{10}	$x_0 = x_1 = x_2 = x_3 = x_4 = x_5 = 0$
f_{11}	$q_4(x_4, x_5, x_6) = 0, x_0 = x_1 = x_2 = x_3 = 0$
f_{12}	$x_0 = x_1 = x_2 = x_3 = x_4 = 0$
f_{13}	$q_4(x_4, x_5) = 0, x_0 = x_1 = x_2 = x_3 = 0$
f_{14}	$x_0 = x_1 = x_2 = x_3 = x_4 = x_5 = 0$
f_{15}	$q(x_5, x_6) = 0, x_0 = x_1 = x_2 = x_3 = x_4 = 0$
f_{16}	$x_0 = x_1 = x_2 = x_3 = x_4 = 0$
f_{17}	$x_0 = x_1 = x_2 = x_3 = x_4 = x_5 = 0$
f_{18}	$x_0 = x_1 = x_2 = x_3 = x_4 = 0$
f_{19}	$x_0 = x_1 = x_2 = x_3 = x_4 = 0$
f_{20}	$x_0 = x_1 = x_2 = x_3 = x_4 = 0$
f_{21}	$x_0 = x_1 = x_2 = x_3 = x_4 = x_5 = 0$
f_{22}	$x_0 = x_1 = x_2 = x_3 = x_4 = x_5 = 0$

4 19 maximal strictly semi-stable cubic fivefolds under the action of $\mathrm{SL}(7)$

For an element σ in $\mathrm{SL}(7)$ and $\mathbb{J} \subseteq \mathbb{I}$, we set $\mathbb{J}^\sigma = \cup_f \mathrm{Supp}(f^\sigma)$, where f runs through all polynomials with $\mathrm{Supp}(f) \subseteq \mathbb{J}$.

Definition 4.1. *We denote*

$$\mathbb{I}(\mathbf{r}_k)_{\geq 0} \subseteq \mathbb{I}(\mathbf{r}_l)_{\geq 0} \bmod \mathrm{SL}(7)$$

when there exists $\sigma \in \mathrm{SL}(7)$ such that $\mathbb{I}(\mathbf{r}_k)_{\geq 0}^\sigma \subseteq \mathbb{I}(\mathbf{r}_l)_{\geq 0}$ and say that $\mathbb{I}(\mathbf{r}_k)_{\geq 0}$ is included in $\mathbb{I}(\mathbf{r}_l)_{\geq 0}$ modulo $\mathrm{SL}(7)$.

We construct a smaller subset \mathcal{M}' of \mathcal{M} such that (1) any element $\mathbb{I}(\mathbf{r}_k)_{\geq 0}$ in \mathcal{M} is included in some element $\mathbb{I}(\mathbf{r}_l)_{\geq 0}$ in $\mathcal{M}' \bmod \mathrm{SL}(7)$, (2) any element $\mathbb{I}(\mathbf{r}_k)_{\geq 0}$ in \mathcal{M}' is not included in other $\mathbb{I}(\mathbf{r}_l)_{\geq 0}$ in $\mathcal{M}' \bmod \mathrm{SL}(7)$ ($1 \leq l \leq 23$)¹.

¹Since $\mathbb{I}(\mathbf{r}_{23})_{\geq 0}$ is also maximal in the set $\{\mathbb{I}(\mathbf{r})_{\geq 0} | \mathbf{r} \in \mathbb{Z}_{(0)}^7\}$, $\mathbb{I}(\mathbf{r}_k)_{\geq 0} \in \mathcal{M}$ may be included in $\mathbb{I}(\mathbf{r}_{23})_{\geq 0}$ modulo $\mathrm{SL}(7)$.

Theorem 4.2. \mathcal{M}' consists of the following 19 elements. As a consequence, there are just 19 irreducible components in the moduli space of strictly semi-stable cubic fivefolds.

- $\mathbb{I}(\mathbf{r}_6)_{\geq 0}, \mathbb{I}(\mathbf{r}_8)_{\geq 0}, \mathbb{I}(\mathbf{r}_{10})_{\geq 0}, \mathbb{I}(\mathbf{r}_{17})_{\geq 0}, \mathbb{I}(\mathbf{r}_{21})_{\geq 0}$ (Singular locus is a point)
- $\mathbb{I}(\mathbf{r}_5)_{\geq 0}, \mathbb{I}(\mathbf{r}_7)_{\geq 0}, \mathbb{I}(\mathbf{r}_{12})_{\geq 0}, \mathbb{I}(\mathbf{r}_{16})_{\geq 0}, \mathbb{I}(\mathbf{r}_{18})_{\geq 0}, \mathbb{I}(\mathbf{r}_{19})_{\geq 0}, \mathbb{I}(\mathbf{r}_{20})_{\geq 0}$ (Singular locus is a line)
- $\mathbb{I}(\mathbf{r}_1)_{\geq 0}, \mathbb{I}(\mathbf{r}_9)_{\geq 0}, \mathbb{I}(\mathbf{r}_{11})_{\geq 0}$ (Singular locus is a conic)
- $\mathbb{I}(\mathbf{r}_{13})_{\geq 0}$ (Singular locus is two lines which intersect at one point)
- $\mathbb{I}(\mathbf{r}_2)_{\geq 0}, \mathbb{I}(\mathbf{r}_3)_{\geq 0}, \mathbb{I}(\mathbf{r}_4)_{\geq 0}$ (Singular locus is a $(2, 2)$ -intersection in \mathbb{P}^3)

We prove this theorem by the following propositions. We first prove including relations. We prepare a notation.

Definition 4.3. Let f, g be two elements of $\mathbb{C}[x_0, \dots, x_6]_3$. We denote

$$f \triangleright g$$

if there exists $\sigma \in \mathrm{SL}(7)$ such that $\mathrm{Supp}(f^\sigma) \subseteq \mathrm{Supp}(g)$.

Proposition 4.4. There are following including relations modulo $\mathrm{SL}(7)$ among the elements of the set $\{\mathbb{I}(\mathbf{r}_1)_{\geq 0}, \dots, \mathbb{I}(\mathbf{r}_{23})_{\geq 0}\}$.

- (1) $\mathbb{I}(\mathbf{r}_{22})_{\geq 0} \subseteq \mathbb{I}(\mathbf{r}_{21})_{\geq 0} \bmod \mathrm{SL}(7)$
- (2) $\mathbb{I}(\mathbf{r}_{15})_{\geq 0} \subseteq \mathbb{I}(\mathbf{r}_{21})_{\geq 0} \bmod \mathrm{SL}(7)$
- (3) $\mathbb{I}(\mathbf{r}_{14})_{\geq 0} \subseteq \mathbb{I}(\mathbf{r}_{21})_{\geq 0} \bmod \mathrm{SL}(7)$
- (4) $\mathbb{I}(\mathbf{r}_{23})_{\geq 0} \subseteq \mathbb{I}(\mathbf{r}_{12})_{\geq 0} \bmod \mathrm{SL}(7)$

Proof. The case (1). We show that generic polynomial f_{22} can be of modified to type f_{21} by finite steps of linear transformation.

$$\begin{aligned} f_{22} &= c(x_0, \dots, x_5) + x_0 l(x_0, \dots, x_5) x_6 \\ &\triangleright c(x_0, \dots, x_5) + x_0 l(x_0, x_1) x_6 \\ &\triangleright c(x_0, \dots, x_5) + q(x_0, x_1) x_6 \\ &= f_{21} \end{aligned}$$

The case (2). Similarly, we have the following.

$$\begin{aligned} f_{15} &= c(x_0, \dots, x_4) + x_0 \{l_1(x_0, \dots, x_4) x_5 + l_2(x_0, \dots, x_4) x_6 + q(x_5, x_6)\} \\ &\triangleright c(x_0, \dots, x_5) + x_0 l(x_0, \dots, x_4) x_6 + x_0 (\alpha_1 x_5 x_6 + \alpha_2 x_6^2) \\ &\triangleright c(x_0, \dots, x_5) + x_0 l(x_0, \dots, x_4) l(x_5, x_6) + x_0 x_5 x_6 \end{aligned}$$

$$\begin{aligned}
& \triangleright c(x_0, \dots, x_5) + x_0 l(x_0, \dots, x_5) x_6 \\
& = f_{22} \triangleright f_{21}
\end{aligned}$$

The case (3).

$$\begin{aligned}
f_{14} &= c(x_0, \dots, x_3) + (q_1(x_0, \dots, x_2) + l(x_0, x_1, x_2)x_3)x_4 + (q_2(x_0, \dots, x_3) + \\
& l(x_0, x_1, x_2)x_3)x_5 + q_3(x_0, x_1, x_2)x_6 + x_0 q_4(x_4, x_5) + x_1 q_5(x_4, x_5) + x_2 q_6(x_4, x_5) \\
& \triangleright c(x_0, \dots, x_5) + q_2(x_0, x_1, x_2)x_5 + q_3(x_0, x_1, x_2)x_6 \\
& \triangleright c(x_0, \dots, x_5) + q_2(x_0, x_1, x_2)(x_5 + tx_6) + q_3(x_0, x_1, x_2)x_6 \\
& \triangleright c(x_0, \dots, x_5) + (q_2(x_0, x_1, x_2)t + q_3(x_0, x_1, x_2))x_6
\end{aligned}$$

We choose $t \in \mathbb{C}$ so that the rank of the quadratic form $(q_2(x_0, x_1, x_2)t + q_3(x_0, x_1, x_2))$ is 2.

$$\begin{aligned}
& \triangleright c(x_0, \dots, x_5) + q(x_0, x_1)x_6 \\
& = f_{21}
\end{aligned}$$

The case (4).

$$\begin{aligned}
f_{23} &= c(x_0, \dots, x_3) + q_1(x_0, \dots, x_3)x_4 + q_2(x_0, \dots, x_3)x_5 + (q_3(x_0, x_1) + \\
& x_0 l(x_2, x_3))x_6 + x_0 q_4(x_4, x_5) \\
& \triangleright c(x_0, \dots, x_3) + q_1(x_0, \dots, x_3)x_4 + q_2(x_0, \dots, x_3)x_5 + (q_3(x_0, x_1) + x_0 l(x_2, x_3))x_6 + \\
& x_0(\alpha_1 x_4^2 + \alpha_2 x_4 x_5) \\
& \triangleright c(x_0, \dots, x_3) + q_1(x_0, \dots, x_3)x_4 + q_2(x_0, \dots, x_3)x_5 + (q_3(x_0, x_1) + x_0 l(x_2, x_3))x_6 + \\
& x_0(\alpha_1 x_4^2 + \alpha_2 x_4 x_5) + \alpha_3 x_1 x_4^2 \\
& = f_{12}
\end{aligned}$$

□

Remark 4.5. By Proposition 4.4 (4), we need not check that $\mathbb{I}(\mathbf{r}_k)_{\geq 0} \subseteq \mathbb{I}(\mathbf{r}_{23})_{\geq 0} \bmod \text{SL}(7)$ for $\mathbb{I}(\mathbf{r}_k)_{\geq 0} \in \mathcal{M}$.

Next, we show that among the 19 elements of \mathcal{M} , there are no including relations. But if we write all proofs, it becomes too long. Since the arguments are very similar, we write the proofs in only two cases.

Proposition 4.6. $\mathbb{I}(\mathbf{r}_5)_{\geq 0}$ is not included in $\mathbb{I}(\mathbf{r}_7)_{\geq 0} \bmod \text{SL}(7)$

Proof. It is impossible to permute the variables x_0, \dots, x_6 because of types of f_5 and f_7 . We compare quadratic forms which are coefficients of x_6 of f_5 and f_7 . They are $Q_5 = q_3(x_0, x_1) + l(x_0, x_1)x_2 + \alpha_2 x_0 x_3$ and $Q_7 = q_3(x_0, x_1) + x_0 l(x_2, x_3)$ respectively. The monomial $x_1 x_2$ of Q_5 does not appear in Q_7 . To delete this monomial, we need linear transformation of x_0, x_1 such that $l(x_0, x_1)x_2$ of Q_5 becomes $x_0 x_2$. By this transformation, the coefficient of $x_1 x_4 x_5$ in f_5 does not vanish. But that of f_7 is zero. □

Proposition 4.7. $\mathbb{I}(\mathbf{r}_7)_{\geq 0}$ is not included in $\mathbb{I}(\mathbf{r}_5)_{\geq 0} \bmod \mathrm{SL}(7)$

Proof. As above, it is impossible to permute the variables x_0, \dots, x_6 . The quadratic forms as coefficients of x_4, x_5 of f_7 are $q_1(x_0, \dots, x_3)$ and $q_2(x_0, x_1, x_2) + l(x_0, x_1)x_3$ respectively. Considering the ranks of two quadratic forms as coefficients of x_4, x_5 of f_5 , it needs linear transformation of the variables x_4, x_5 in f_7 . But if we apply such linear transformation the coefficient of x_5^2 does not vanish. But that of f_5 is zero. \square

Finally, we show that these 19 elements are semi-stable in the sense of GIT (i.e. not with respect to \mathbb{T}).

Proposition 4.8. Any cubic fivefold defined by a generic polynomial f_k whose support is $\mathbb{I}(\mathbf{r}_k)_{\geq 0} \in \mathcal{M}'$ is strictly semi-stable.

Proof. We use the numerical criterion Theorem 1.2. For any $\mathbf{r} \in \mathbb{Z}_{(0)}^7$, $\mathbb{I}(\mathbf{r})_{\geq 0}$ is not stable with respect to \mathbb{T} . Let f_k be a generic polynomial whose support is $\mathbb{I}(\mathbf{r}_k)_{\geq 0} \in \mathcal{M}'$. We show that f_k^σ is semi-stable with respect to \mathbb{T} for any $\sigma \in \mathrm{SL}(7)$. If f_k^σ is not semi-stable with respect to \mathbb{T} (i.e. unstable with respect to \mathbb{T}) for some $\sigma \in \mathrm{SL}(7)$, then some $\mathbb{I}(\mathbf{r}_l)_{\geq 0} \in \mathcal{M}$ contains $\mathrm{Supp}(f_k^\sigma)$. This fact contradicts to the property (2) of \mathcal{M}' . \square

5 Main theorems

The following theorem is a direct consequence of Theorem 4.2.

Theorem 5.1. The moduli space of stable cubic fivefolds are compactified by adding 19 irreducible components. The singular locus of the cubic fivefold corresponding to the generic point of an irreducible component contains one of the following²:

- a point whose multiplicity is equals or greater than 15.
- a line whose multiplicity is 1 or 2.
- two lines which intersect at one point whose multiplicities are 1.
- a conic whose multiplicity is 1 or 2.
- a (2, 2)-intersection in \mathbb{P}^3 whose multiplicity is 1 or 2.

Corollary 5.2. Let X be a cubic fivefold. We assume that the any component of singular locus of X is a point. If degree of each point is less than 15, X is stable.

²If the cubic fivefold is special, the singular locus contains degenerated one of the following:

References

- [1] Clemens C.H. and Griffiths P.A., The Intermediate Jacobian of the Cubic Threefold, The Annals of Mathematics, Second Series, Vol. 95, No. 2 (Mar., 1972), 281-356.
- [2] Collino A., The Abel-Jacobi isomorphism for the cubic fivefold, Pacific J. Math., vol.122, No1, (1986), 43-55
- [3] Cox D., Little J., O'shea D., Ideals, varieties, and algorithms, Springer, New York. 2007.
- [4] Dolgachev I., Lectures on Invariant Theory, London Mathematical Society Lecture Note Series 296. Cambridge University Press, Cambridge, 2003.
- [5] Edelsbrunner H., Algorithms in combinatorial geometry. EATCS Monographs on Theoretical Computer Science, 10. Springer-Verlag, Berlin. 1987.
- [6] Hilbert D., Über die vollen Invariantensysteme, Mathem. Annalen 42(1893), 313-373.
- [7] Laza R., the moduli space of cubic fourfolds, J.Algebraic Geom. 18 (2009), 511-545
- [8] Mumford D., Fogarty J., Kirwan F., Geometric Invariant Theory, Third edition. Springer-Verlag, Berlin, 1982.
- [9] Yokoyama M., Stability of cubic 3-fold, Tokyo J.Math. 25(2002), 85-105
- [10] Yokoyama M., Stability of cubic hypersurfaces of dimension. Higher dimensional algebraic varieties and vector bundles, RIMS Kokyuroku Bessatsu, B9, Res. Inst. Math. Sci. (RIMS), Kyoto (2008), 189-204